ANALYSIS OF FRACTIONAL ORDER HEAT CONDUCTION PROBLEMS IN NONCYLINDRICAL DOMAINS WITH DIFFERENT BOUNDARY CONDITIONS

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Abstract. *The present article concerning the mathematical modelling of thermophysical processes in the electric arcs of high-current tripping devices. The heat conduction equation, which takes into account the influence of thermal sources in the arc and the effect of shrinkage of the axial section of the arc in the cathode region into a contact, is one of the tools for describing the physics of processes in the arc. The contacts are in a closed state at the start of time, and there is no domain of problem solution. From a mathematical point of view, the problematicity of the problem at hand is precisely the presence of a moving boundary and the degeneracy of the solution domain at the outset.*

Keywords: *Fractional derivative, Heat conduction, Noncylindrical Domains, Mittag-Leffler function.*

1. INTRODUCTION

Heat conduction was defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones. It was stated that conduction can take place in liquids and gases as well as solids provided that there is no bulk motion involved.

Caputo [1] developed the fractional order derivatives of the differential equations and it's filters. Povstenko [2,4] solved the Boundary value problems using the fractional order derivatives, in an infinite medium with a spherical inclusion. Ankhmanova et at. [3] Solved a singular integral equation of the voltera type and it's adjoint. Jenaliyev et at. [5], studied boundary value problems of the heat equation in noncylindrical domains degenerating at the initial moment leads to the necessity of research of the singular Volterra integral equations of the second kind, when the norm of the integral operator is equal to 1. The paper deals with the singular Volterra integral equation of the second kind, to which by virtue of ‘the incompressibility’ of the kernel the classical method of successive approximations is not applicable.

Amangaliyeva et at. [6, 7] established that in an infinite angular domain for Dirichlet problem of the heat conduction equation the unique (up to a constant factor) non-trivial solution exists, which does not belong to the class of summable functions with the found weight. It is shown that for the adjoint boundary value problem the unique (up to a constant factor) non-trivial solution exists, which belongs to the class of essentially bounded functions with the weight found in the work. It is proved that the operator of a boundary value problem of heat conductivity in an infinite angular domain in a class of growing functions is Noetherian with an index which is equal to minus one. Dzhenaliyec et at. [8] solved the boundary value problems of heat conduction equation in an unbounded plane. Recently many of fractional order heat conduction problems have been discussed [9-16].

There are presently issues concerning the mathematical modelling of thermophysical processes in the electric arcs of high-current tripping devices. The heat conduction equation, which takes into account the influence of thermal sources in the arc and the effect of shrinkage of the axial section of the arc in the cathode region into a contact, is one of the tools for describing the physics of processes in the arc.

The contacts are in a closed state at the start of time, and there is no domain of problem solution. From a mathematical point, the problematicity of the problem at hand is precisely the presence of a moving boundary and the degeneracy of the solution domain at the outset.

2. BASIC DEFINITIONS

- **Riemann Liouville Fractional Integral:**

The Riemnn-Liouville fractional integral defined as:

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt $$

Where $\alpha > 0$ is any non-negative real number, *f(x)* be piecewise continuous on *(a, x)* and integrable on any finite subinterval of *[a, x]*.

- **Riemann Liouville Fractional Derivative:**

The fractional derivative can be defined using the definition of the fractional integral. Suppose that on $n-1 < \alpha \le n$ and n is the smallest integer greater than on α. Then the fractional derivative of *f(x)* of order $\alpha > 0$ is:

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\,dt $$

- **Caputo's Fractional Derivative:**

The Caputo's fractional derivative of *f(x)* of order $\alpha > 0$, is

$$ {}_a^CD_x^{-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t)\,dt, $$

where $n-1 < \alpha \le n$

3. PROBLEM FORMULATION

We consider BVP's of fractional heat conduction in a degenerate domain.

a) In the domain

$G = \{(x;\, t):\ 0 < x < t,\ t > 0\}$ it is required to find a solution to the equation of fractional heat conduction.

$$ \frac{\partial^\alpha u}{\partial t^\alpha} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1} $$

with the boundary conditions

$$ u(x,t)\,|_{x=0} = 0,\ u(x,t)\,|_{x=t} = 0 \tag{2} $$

b) In the domain

$G = \{(x;\, t):\ 0 < x < t, t > 0\}$ it is required to find a solution to the equation of fractional heat conduction.

$$ \frac{\partial^\alpha u}{\partial t^\alpha} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{3} $$

with the boundary conditions

$$ u_x(x,t)\,|_{x=0} = 0,\ u_x(x,t)\,|_{x=t} = 0 \tag{4} $$

c) In the domain

$G = \{(x;\, t):\ 0 < x < t, t > 0\}$ it is required to find a solution to the equation of fractional heat conduction.

$$ \frac{\partial^\alpha u}{\partial t^\alpha} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{5} $$

with the boundary conditions

$$ u(x,t)\,|_{x=0} = 0,\ u_x(x,t)\,|_{x=t} = 0 \tag{6} $$

d) In the domain

$G = \{(x;\, t):\ 0 < x < t, t > 0\}$ it is required to find a solution to the equation of fractional heat conduction.

$$ \frac{\partial^\alpha u}{\partial t^\alpha} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{7} $$

with the boundary conditions

$$ u_x(x,t)\,|_{x=0} = 0,\ u(x,t)\,|_{x=t} = 0 \tag{8} $$

4.CONVERSION OF PROBLEMS IN INTEGRAL EQUATIONS

We consider a solution of the Problem (a) as the sum of the thermal potentials of the double layer

$$ \begin{aligned} u(x,t) &= \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x}{(\sqrt{t-\tau})^3} x^\beta \\ &\times E_{\alpha\beta}(-4a^2(t-\tau)^\alpha)\nu(\tau)\,d\tau \\ &+ \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x-\tau}{(\sqrt{t-\tau})^3} (x-\tau)^\beta \\ &\times E_{\alpha\beta}(-4a^2(t-\tau)^\alpha)\varphi(\tau)\,d\tau. \end{aligned} \tag{9} $$

where $E_{\alpha\ \beta}(.)$ is Mittag-Leffler function of two parameters.

We consider a solution of the Problem (b) as the sum of the thermal potentials of the simple layer

$$ \begin{aligned} u(x,t) &= \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} x^\beta \\ &\times E_{\alpha\beta}(-4a^2(t-\tau)^\alpha)\nu(\tau)\,d\tau \\ &+ \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} (x-\tau)^\beta \\ &\times E_{\alpha\beta}(-4a^2(t-\tau)^\alpha)\varphi(\tau)\,d\tau. \end{aligned} \tag{10} $$

We consider a solution of the Problem (c) as the sum of a combination of the thermal potentials of a double and a simple layer

$$u(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x}{\left(\sqrt{t-\tau}\right)^3} x^{\beta}$$
$$\times E_{\alpha\beta}\left(-4a^2 (t-\tau)^{\alpha}\right) \nu(\tau) d\tau$$
$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} (x-\tau)^{\beta} \tag{11}$$
$$\times E_{\alpha\beta}\left(-4a^2 (t-\tau)^{\alpha}\right) \varphi(\tau) d\tau.$$

We consider a solution of the Problem (d) as the sum of a combination of the thermal potentials of a simple and a double layer

$$u(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} x^{\beta}$$
$$\times E_{\alpha\beta}\left(-4a^2 (t-\tau)^{\alpha}\right) \nu(\tau) d\tau$$
$$+ \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x-\tau}{\left(\sqrt{t-\tau}\right)^3} (x-\tau)^{\beta} \tag{12}$$
$$\times E_{\alpha\beta}\left(-4a^2 (t-\tau)^{\alpha}\right) \varphi(\tau) d\tau.$$

It is clear that the functions (9)-(12) satisfy the heat equation for any $\nu(t)$ and $\varphi(t)$ [20].

$$\varphi(t) - \int_0^t K(t,\tau)\, \varphi(\tau) d\tau = 0 \tag{13}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \begin{array}{c} \frac{t+\tau}{\left(\sqrt{t-\tau}\right)^3} (t+\tau)^{\beta} \left(E_{\alpha\beta}\left(-4a^2 (t-\tau)^{\alpha}\right) \right) \\ + \frac{1}{\sqrt{t-\tau}} \left((t-\tau)^{\beta - \frac{1}{2}} E_{\alpha\beta}\left(-4a^2 t^{\alpha}\right) \right) \end{array} \right\}$$

Problem (c) and (d) are reduced to the integral equation:

$$\varphi(t) - \int_0^t K(t,\tau)\, \varphi(\tau) d\tau = 0 \tag{14}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \begin{array}{c} -\frac{t+\tau}{\left(\sqrt{t-\tau}\right)^3} exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) \\ + \frac{1}{\sqrt{t-\tau}} exp\left(-\frac{t-\tau}{4a^2}\right) \end{array} \right\}$$

Singularity of the kernel $K(t,\tau)$ of equation (13) is determined by the properties

$$\lim_{t\to 0} \int_0^t K(t,\tau) d\tau = 1, \quad \lim_{t\to +\infty} \int_0^t K(t,\tau) d\tau = 1. \tag{15}$$

By the Carleman-Vekua method, solving the integral equation (13) is reduced to solving a nonhomogeneous Abel equation. The boundary value problems (a) and (b) are studied weight spaces, and the classes of uniqueness for their solutions.

The following theorem is proved.

Theorem 1:

The function

$$\varphi(t) = \frac{1}{\sqrt{t}} exp\left(-\frac{t}{4a^2}\right) + \frac{\sqrt{\pi}}{2a} erf\left(\frac{\sqrt{t}}{2a}\right) + \frac{\sqrt{\pi}}{2a}$$

where

$$erf\ z = \frac{2}{\sqrt{\pi}} \int_0^z exp(-\zeta^2) d\zeta$$

is a solution of the integral equation (13) in the weight class of functions

$$\sqrt{t}\, exp\left(-\frac{t}{4a^2}\right) \varphi(t) \in L_{\infty}(0,\infty).$$

Singularity of the kernel $K(t,\tau)$ of equation (14) is determined by the properties

$$\lim_{t\to 0} \left| \int_0^t K(t,\tau) d\tau \right| = 1, \quad \lim_{t\to +\infty} \left| \int_0^t K(t,\tau) d\tau \right| = 1. \tag{16}$$

In fact, for the kernel $K(t,\tau)$ of equation (14) making a substitution $\sqrt{t-\tau}$ we obtain

$$\int_0^t K(t,\tau) d\tau = -exp\left(\frac{2t}{a^2}\right) erfc\left(\frac{3\sqrt{t}}{2a}\right) + erf\left(\frac{\sqrt{t}}{2a}\right).$$

4. CONCLUDING REMARK

The singularity of the obtained integral equations lies in the equation (15) and (16) of the corresponding kernel $K(t,\tau)$ and this singularity is expressed in the fact that the corresponding nonhomogeneous equations cannot be solved by the method of successive approximations. Equations (15) and (16) indicate the "incompressibility" of the kernel of integral equations. The results presented here will be more useful in studying the heat conduction problem in noncylindrical bodies in real-life engineering problems, mathematical biology by considering the fractional derivative in the field equations.

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